

COMMUTING HIGHER RANK ORDINARY DIFFERENTIAL OPERATORS

ANDREY E. MIRONOV

ABSTRACT. The theory of commuting ordinary differential operators was developed from the beginning of the XX century on. The problem of finding commuting differential operators is solved for the case of operators of rank one. For operators of rank greater than one it is still open. In this paper we discuss some results related to operators of rank greater than one.

1. INTRODUCTION

The commutativity condition $L_1L_2 = L_2L_1$ of two differential operators

$$L_n = \partial_x^n + \sum_{i=0}^{n-2} u_i(x) \partial_x^i, \quad L_m = \partial_x^m + \sum_{i=0}^{m-2} v_i(x) \partial_x^i$$

is equivalent to a complicated system of nonlinear differential equation on coefficients $u_i(x), v_i(x)$. The theory of commuting ordinary differential operators was first developed in the beginning of the XX century in the works of Wallenberg [1], Schur [2], and Burchnell, Chaundy [3]. Let me recall these classical results.

Wallenberg [1] found operators for the case of $n = 2, m = 3$

$$(1) \quad L_1 = \partial_x^2 + u(x), \quad L_2 = \partial_x^3 + \left(\frac{s_2}{4} + \frac{3}{2}u(x) \right) \partial_x + \frac{3}{4}u'(x),$$

where $u(x)$ satisfies the integrable equation

$$(u')^2 + 2u^3 + s_2u^2 + s_1u + s_0 = 0, \quad s_i \in \mathbb{C}.$$

Let L_k be an operator of order $k \geq 1$. Schur [2] proved the following theorem.

Theorem 1.1. *If $L_nL_k = L_kL_n$ and $L_mL_k = L_kL_m$, then $L_nL_m = L_mL_n$.*

To prove this theorem Schur used pseudo-differential operators. There is a pseudo-differential operator $S = 1 + s_1(x)\partial_x^{-1} + s_2(x)\partial_x^{-2} + \dots$ such that $\hat{L}_k = S^{-1}L_kS = \partial_x^k$. Pseudo-differential operator $\hat{L}_n = S^{-1}L_nS$ commutes with $\hat{L}_k = \partial_x^k$. This is possible only if \hat{L}_n has constant coefficients. Similarly, $\hat{L}_m = S^{-1}L_mS$ has constant coefficients. This means that $\hat{L}_n\hat{L}_m = \hat{L}_m\hat{L}_n$ and hence $L_nL_m = L_mL_n$.

The next theorem was proved by Burchnell and Chaundy [3].

Theorem 1.2. *If $L_nL_m = L_mL_n$, then there is a nonzero polynomial $R(z, w)$ such that $R(L_n, L_m) = 0$.*

The author is grateful to the Hausdorff Research Institute for Mathematics (Bonn) for hospitality during the writing of this paper. The work was also partially supported by a grant MD-5134.2012.1 from the President of Russia; and by a grant from Dmitri Zimin's "Dynasty" foundation.

For example, for Wallenberg's operators the polynomial R has the form

$$R(z, w) = w^2 - \left(z^3 + \frac{s_2}{2} z^2 + \frac{s_2^2 + 2s_1}{16} z + \frac{s_1 s_2 - 2s_0}{32} \right).$$

The curve $\Gamma = \{(z, w) \in \mathbb{C}^2 : R(z, w) = 0\}$ is called the *spectral curve*. The spectral curve parametrizes common eigenvalues of L_n and L_m . If

$$L_n \psi = z \psi, \quad L_m \psi = w \psi,$$

then $(z, w) \in \Gamma$. The dimension l of the space of common eigenfunctions of L_n, L_m for fixed z, w is called the *rank*. The number l is the same for general $P = (z, w) \in \Gamma$. The rank equals the greatest common divisor of n and m .

Commutative rings of ordinary differential operators were classified by Krichever [4], [5]. The ring is determined by spectral data. If the rank is one, then the spectral data define commuting operators by explicit formulas (see [4]). In the case of operators of rank greater than one there are the following results. Krichever and Novikov [6], [7] found operators of rank two corresponding to elliptic spectral curves. Mokhov [8] found operators of rank three also corresponding to elliptic spectral curves. There are also examples of operators of rank greater than one corresponding to spectral curves of genus $g > 1$ (see [9]–[13]).

In Section 2 we recall the construction of rank one operators and give several examples. In Section 3 we recall the method of deformations of Tyurin parameters and some results about operators of rank two corresponding to elliptic spectral curves. Self-adjoint operators of rank two corresponding to hyperelliptic spectral curves of arbitrary genus are considered in Section 4. Some open problems related to operators of rank greater than one are discussed in Section 5.

2. COMMUTING DIFFERENTIAL OPERATORS OF RANK ONE

Let the spectral curve Γ be a compact Riemann surface. In the case of operators of rank one common eigenfunction $\psi(x, P), P \in \Gamma$ of L_n and L_m has the following properties.

- 1) Function ψ has one essential singularity at a fixed point $q \in \Gamma$

$$\psi = e^{kx} \left(1 + \frac{\xi_1(x)}{k} + \frac{\xi_2(x)}{k^2} + \dots \right),$$

where k^{-1} is a local parameter in a neighbourhood of q .

- 2) Function ψ has simple poles at some points $\gamma_1, \dots, \gamma_g$, where g is the genus of Γ .

The set $\{\Gamma, q, k^{-1}, \gamma_1, \dots, \gamma_g\}$ is called the *spectral data*. If we take the spectral data where $\gamma_1 + \dots + \gamma_g$ is a non-special divisor, then there is a unique function $\psi(x, P)$ satisfying the conditions 1) and 2). The function $\psi(x, P)$ is called the *Baker–Akhiezer function*. Let us take a meromorphic function $f(P)$ on Γ with the unique pole of order n at q . There is a differential operator $L(f)$ of order n such that

$$L(f)\psi(x, P) - f(P)\psi(x, P) = (\partial_x^n + u_{n-2}(x)\partial_x^{n-2} + \dots + u_0(x))\psi(x, P) = e^{xk}O(1/k).$$

Coefficients $u_i(x)$ depend on $\xi_i(x)$. If the right hand side of the last equality is not zero, then we can add it to the Baker–Akhiezer function and get a new function which has the same properties 1) and 2). But this is impossible, because the Baker–Akhiezer function is unique. Hence, $L(f)\psi = f(P)\psi$. Similarly, for the

meromorphic function $g(P)$ with the unique pole of order m at q we have $L(g)\psi = g(P)\psi$. Operators $L(f)$ and $L(g)$ commute because the commutator has an infinite dimensional kernel $[L(f), L(g)]\psi(x, P) = 0$. So, every commutative ring of operators of rank one with the nonsingular spectral curve corresponds to the spectral data, and the spectral data define a commutative ring of ordinary differential operators.

The Baker–Akhiezer function can be expressed via the theta-function of the Jacobi variety of Γ . Let us choose a basis of circles $a_i, b_i, 1 \leq i \leq g$ on Γ with the indices of intersections:

$$a_i \circ a_j = 0, \quad b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij}.$$

Let $\omega_1, \dots, \omega_g$ be a normalized basis of Abelian differentials $\int_{a_i} \omega_j = \delta_{ij}$. The theta-function of the Jacobi variety

$$J(\Gamma) = \mathbb{C}^g / \{\mathbb{Z}^g + \Omega\mathbb{Z}^g\}$$

is given by the series

$$\theta(z) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i \langle \Omega n, n \rangle + 2\pi i \langle n, z \rangle), \quad z \in \mathbb{C}^g,$$

where $\langle n, z \rangle = n_1 z_1 + \dots + n_g z_g$, the matrix Ω has the components

$$\Omega_{ij} = \Omega_{ji} = \int_{b_i} \omega_j.$$

The theta-function has the property

$$\theta(z + \Omega m + n) = \exp(-\pi i \langle \Omega m, m \rangle - 2\pi i \langle m, z \rangle) \theta(z), \quad m, n \in \mathbb{Z}^g.$$

Let ω be a meromorphic one-form on Γ with pole of order two at q . We assume that ω is normalized by the condition $\int_{a_i} \omega = 0$. Let U be a vector of b -periods of ω :

$$U = \left(\int_{b_1} \omega, \dots, \int_{b_g} \omega \right).$$

Denote by $A(P) : \Gamma \rightarrow J(\Gamma)$ the Abel map

$$A(P) = \left(\int_{P_0}^P \omega_1, \dots, \int_{P_0}^P \omega_g \right),$$

P_0 is a fixed point. The function

$$\varphi(x, P) = \frac{\theta(A(P) - A(\gamma_1) - \dots - A(\gamma_g) - K_\Gamma + xU)}{\theta(A(P) - A(\gamma_1) - \dots - A(\gamma_g) - K_\Gamma)} \exp(2\pi i x \int_{P_0}^P \omega),$$

where K_Γ is a vector of Riemann constants is correctly defined on Γ . The function φ has simple poles at $\gamma_1, \dots, \gamma_g$ and has the following form in the neighbourhood of q

$$\varphi = e^{kx} \left(\xi_0(x) + \frac{\xi_1(x)}{k} + \dots \right).$$

After the normalization $\psi(x, P) = \frac{\varphi(x, P)}{\xi_0(x)}$ we get the Baker–Akhiezer function.

Let us consider the following examples.

Example 1. Let Γ be $\bar{\mathbb{C}} = \mathbb{CP}^1$, z — affine coordinate, $q = \infty$. The Baker–Akhiezer function is

$$\psi(x, z) = e^{xz}.$$

On $\Gamma \setminus \{q\}$ the function ψ has no poles because the genus of Γ equals zero. For the meromorphic function

$$f(z) = z^n + c_{n-1}z^{n-1} + \dots + c_0, \quad c_i \in \mathbb{C}$$

with the pole of order n at q we have the operator with the constant coefficients

$$L(f) = \partial_x^n + c_{n-1}\partial_x^{n-1} + \dots + c_0.$$

Example 2. Let $\Gamma = \mathbb{C}/\{2\mathbb{Z}\omega + 2\mathbb{Z}\omega'\}$ be an elliptic spectral curve and $q = 0$. The Baker–Akhiezer function is

$$\psi = e^{-x\zeta(z)} \frac{\sigma(z+x+\gamma)}{\sigma(x+\gamma)\sigma(z+\gamma)}.$$

We have

$$\begin{aligned} L_2\psi &= (\partial_x^2 - 2\wp(x+\gamma))\psi = \wp(z)\psi, \\ L_3\psi &= \left(\partial_x^3 - 3\wp(x+\gamma)\partial_x - \frac{3}{2}\wp'(x+\gamma) \right) \psi = \frac{1}{2}\wp'(z)\psi, \end{aligned}$$

where $\sigma(z), \zeta(z), \wp(z)$ are Weierstrass functions. Operators L_1, L_2 commute and satisfy the equation

$$L_3^2 = L_2^3 - \frac{g_2}{4}L_2 - \frac{g_3}{4},$$

where g_2, g_3 are some constants.

Example 3. Under the degeneration $g_2, g_3 \rightarrow 0$ in the Example 2 we get the caspidal spectral curve. Under this degeneration the functions $\sigma(z), \zeta(z), \wp(z)$ become

$$\hat{\sigma}(z) = z, \quad \hat{\zeta}(z) = \frac{1}{z}, \quad \hat{\wp}(z) = \frac{1}{z^2}.$$

We get commuting differential operators with rational coefficients

$$\begin{aligned} \hat{\psi}(x, z) &= e^{-\frac{x}{z}} \frac{z+x+\gamma}{(x+\gamma)(z+\gamma)}, \\ \hat{L}_2\hat{\psi} &= \left(\partial_x^2 - \frac{2}{(x+\gamma)^2} \right) \hat{\psi} = \frac{1}{z^2}\hat{\psi}, \\ \hat{L}_3\hat{\psi} &= \left(\partial_x^3 - \frac{3}{(x+\gamma)^2}\partial_x + \frac{3}{(x+\gamma)^3} \right) \hat{\psi} = -\frac{1}{z^3}\hat{\psi}, \\ \hat{L}_2^3 &= \hat{L}_3^2. \end{aligned}$$

Example 4. Let us consider another degeneration of the elliptic spectral curve — the sphere with one double point. We identify two points on $\bar{\mathbb{C}} = \mathbb{CP}^1$, $\Gamma = \mathbb{CP}^1/\{a \sim -a\}$. In the case of singular spectral curve the definition of the Baker–Akhiezer function is the same, but we should replace the genus by the arithmetic genus of the singular spectral curve. The arithmetic genus of Γ is one. The Baker–Akhiezer function has the form

$$(2) \quad \psi(x, z) = e^{xz} \left(1 + \frac{\xi(x)}{z-\gamma} \right), \quad q = \infty.$$

From the identity

$$(3) \quad \psi(x, a) = \psi(x, -a)$$

we find

$$\xi(x) = \frac{(\gamma^2 - a^2) \sinh(ax)}{a \cosh(ax) + \gamma \sinh(ax)}.$$

The functions $f(z) = z^2$, $g(z) = z^3 - a^2z$ are rational functions on Γ with the poles of order 2 and 3 at q . Thus we have

$$(4) \quad L(f)\psi = (\partial_x^2 + u(x))\psi = z^2\psi,$$

$$(5) \quad L(g)\psi = \left(\partial_x^3 + \left(\frac{3}{2}u(x) - a^2 \right) \partial_x + \frac{3}{4}u'(x) \right) \psi = (z^3 - a^2z)\psi,$$

$$u(x) = \frac{2a^2(a^2 - \gamma^2)}{(a \cosh(ax) + \gamma \sinh(ax))^2}.$$

The Burchnell–Chaundy polynomial of L_1, L_2 is

$$(6) \quad F(\lambda, \mu) = \lambda^2 - \mu(\mu - a^2)^2.$$

Example 5. If the spectral curve is singular, then in general ψ is not a function on the spectral curve, but ψ is a section of a torsion-free sheaf on $\Gamma \setminus \{q\}$ (see [14]). Let us consider one example. We take the same spectral curve as in the Example 4, ψ has the form (2), but instead of (3) we require

$$\psi(x, a) = b\psi(x, -a), \quad b \in \mathbb{C}^*.$$

In this case ψ is a section of a sheaf. We have (4)–(6), where

$$u(x) = \frac{8a^2b(a^2 - \gamma^2)}{((a + ab + \gamma - b\gamma) \cosh(ax) + (a - ab + \gamma + b\gamma) \sinh(ax))^2}.$$

3. METHOD OF DEFORMATION OF TYURIN PARAMETERS

Let me recall the definition of the Baker–Akhiezer function at $l > 1$ (see [5]). We take the *spectral data*

$$\{\Gamma, q, k^{-1}, \gamma, \alpha, \omega(x)\},$$

where Γ is a Riemann surface of genus g , q is a fixed point on Γ , k^{-1} is a local parameter near q ,

$$\omega(x) = (\omega_1(x), \dots, \omega_{l-1}(x))$$

is a set of smooth functions, $\gamma = \gamma_1 + \dots + \gamma_{l_g}$ is a divisor on Γ , α is a set of vectors

$$\alpha_1, \dots, \alpha_{l_g}, \quad \alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,l-1}).$$

The pair (γ, α) is called the *Tyurin parameters*. The Tyurin parameters define a stable holomorphic vector bundle on Γ of rank l and degree lg with holomorphic sections η_1, \dots, η_l . The points $\gamma_1, \dots, \gamma_{l_g}$ are the points of the linear dependence

$$\eta_l(\gamma_i) = \sum_{j=1}^{l-1} \alpha_{j,i} \eta_j(\gamma_i).$$

The vector-function $\psi = (\psi_1, \dots, \psi_l)$ is defined by the following properties.

1. In the neighbourhood of q the vector-function ψ has the form

$$\psi(x, P) = \left(\sum_{s=0}^{\infty} \xi_s(x) k^{-s} \right) \Psi_0(x, k),$$

where $\xi_0 = (1, 0, \dots, 0)$, $\xi_i(x) = (\xi_i^1(x), \dots, \xi_i^l(x))$, the matrix Ψ_0 satisfies the equation

$$\frac{d\Psi_0}{dx} = A\Psi_0, \quad A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ k + \omega_1 & \omega_2 & \omega_3 & \dots & \omega_{l-1} & 0 \end{pmatrix}.$$

2. The components of ψ are meromorphic functions on $\Gamma \setminus \{q\}$ with the simple poles $\gamma_1, \dots, \gamma_{lg}$, and

$$\text{Res}_{\gamma_i} \psi_j = \alpha_{i,j} \text{Res}_{\gamma_i} \psi_l, \quad 1 \leq i \leq lg, \quad 1 \leq j \leq l-1.$$

For the rational function $f(P)$ on Γ with the unique pole of order n at q there is a linear differential operator $L(f)$ of order ln such that $L(f)\psi(x, P) = f(P)\psi(x, P)$. For two such functions $f(P), g(P)$ operators $L(f), L(g)$ commute.

The main difficulty to construct operators of rank $l > 1$ is the fact that the Baker–Akhiezer function is not found explicitly. But operators can be found by the following method of deformation of Tyurin parameters.

The common eigenfunctions of commuting differential operators of rank l satisfy the linear differential equation of order l

$$\psi^{(l)}(x, P) = \chi_0(x, P)\psi(x, P) + \dots + \chi_{l-1}(x, P)\psi^{(l-1)}(x, P).$$

The coefficients χ_i are rational functions on Γ (see [5]) with the simple poles $P_1(x), \dots, P_{lg}(x) \in \Gamma$, and with the following expansions in the neighbourhood of q

$$\chi_0(x, P) = k + g_0(x) + O(k^{-1}),$$

$$\chi_j(x, P) = g_j(x) + O(k^{-1}), \quad 0 < j < l-1,$$

$$\chi_{l-1}(x, P) = O(k^{-1}).$$

Let $k - \gamma_i(x)$ be a local parameter near $P_i(x)$. Then

$$\chi_j = \frac{c_{i,j}(x)}{k - \gamma_i(x)} + d_{i,j}(x) + O(k - \gamma_i(x)).$$

Functions $c_{ij}(x), d_{ij}(x)$ satisfy the following equations (see [5]).

Theorem 3.1.

$$(7) \quad c_{i,l-1}(x) = -\gamma'_i(x),$$

$$(8) \quad d_{i,0}(x) = \alpha_{i,0}(x)\alpha_{i,l-2}(x) + \alpha_{i,0}(x)d_{i,l-1}(x) - \alpha'_{i,0}(x),$$

$$(9) \quad d_{i,j}(x) = \alpha_{i,j}(x)\alpha_{i,l-2}(x) - \alpha_{i,j-1}(x) + \alpha_{i,j}(x)d_{i,l-1}(x) - \alpha'_{i,j}(x), \quad j \geq 1,$$

where $\alpha_{i,j}(x) = \frac{c_{i,j}(x)}{c_{i,l-1}(x)}$, $0 \leq j \leq l-1$, $1 \leq i \leq lg$.

To find χ_i one should solve the equations (7)–(9). Using χ_i one can find coefficients of the operators. At $g = 1$, $l = 2$ Krichever and Novikov [6, 7] solved these equations and found the operators.

Theorem 3.2. *The operator of order 4 has the form*

$$L_{KN} = (\partial_x^2 + u)^2 + 2c_x(\wp(\gamma_2) - \wp(\gamma_1))\partial_x + (c_x(\wp(\gamma_2) - \wp(\gamma_1)))_x - \wp(\gamma_2) - \wp(\gamma_1),$$

where

$$\gamma_1(x) = \gamma_0 + c(x), \quad \gamma_2(x) = \gamma_0 - c(x),$$

$$u(x) = -\frac{1}{4c_x^2} + \frac{1}{4} \frac{c_{xx}^2}{c_x^2} + 2\Phi(\gamma_1, \gamma_2)c_x - \frac{c_{xxx}}{2c_x} + c_x^2(\Phi_c(\gamma_0 + c, \gamma_0 - c) - \Phi^2(\gamma_1, \gamma_2)),$$

$$\Phi(\gamma_1, \gamma_2) = \zeta(\gamma_2 - \gamma_1) + \zeta(\gamma_1) - \zeta(\gamma_2),$$

$\zeta(z), \wp(z)$ are the Weierstrass functions, $c(x)$ is an arbitrary smooth function, γ_0 is a constant.

The operator \tilde{L}_{KN} commuting with L_{KN} can be found from the identity

$$\tilde{L}_{KN}^2 = 4L_{KN}^3 + g_2L_{KN} + g_3.$$

The operators L_{KN}, \tilde{L}_{KN} were studied by many authors (see [15]–[23]).

Dixmier [24] constructed an example of the commutative subalgebras in the first Weyl algebra. Let $A_1 = \mathbb{C}\langle p, q : [p, q] = 1 \rangle$ be the Weyl algebra.

Theorem 3.3. *Two elements of A_1 ,*

$$X = (p^3 + q^2 + h)^2 + 2p,$$

$$Y = (p^3 + q^2 + h)^3 + \frac{3}{2} (p(p^3 + q^2 + h) + (p^3 + q^2 + h)p), \quad h \in \mathbb{C}$$

commute and satisfy the equation $Y^2 = X^3 - h$.

If we substitute $p = x, q = -\partial_x$ in Theorem 3.3 (we can do so, because $[x, -\partial_x] = 1$), then we get operators of rank two

$$L_D = (\partial_x^2 + x^3 + h)^2 + 2x,$$

$$\tilde{L}_D = (\partial_x^2 + x^3 + h)^3 + \frac{3}{2} (x(\partial_x^2 + x^3 + h) + (\partial_x^2 + x^3 + h)x).$$

Operator L_D coincides with L_{KN} for some $c(x)$. Then, a natural question is how to obtain L_D from L_{KN} (Gelfand's problem). The answer is given in the following theorem by Grinevich [16].

Theorem 3.4. *Operator L_{KN} corresponding to the curve $w^2 = 4z^3 + g_2z + g_3$ has rational coefficients if and only if*

$$c(x) = \int_{q(x)}^{\infty} \frac{dt}{\sqrt{4t^3 + g_2t + g_3}},$$

where $q(x)$ is a rational function. If $\gamma_0 = 0$ and $q(x) = x$, then the operator L_{KN} coincides with L_D .

Grinevich and Novikov [15] found conditions when L_{KN} is self-adjoint.

Theorem 3.5. *Operator L_{KN} is self-adjoint if and only if $\wp(\gamma_1) = \wp(\gamma_2)$.*

Spectral properties of operators with periodic coefficients of rank $l > 1$ with a spectral curve of arbitrary genus were studied by Novikov in [25].

4. SELF-ADJOINT OPERATORS OF RANK TWO

If $l = 2$, then

$$(10) \quad \psi'' = \chi_0 \psi + \chi_1 \psi'.$$

In the neighbourhood of q we have the expansions

$$(11) \quad \chi_0 = \frac{1}{k} + a_0(x) + a_1(x)k + O(k^2), \quad \chi_1 = b_1(x)k + b_2(x)k^2 + O(k^3).$$

Functions χ_0, χ_1 have $2g$ simple poles $P_1(x), \dots, P_{2g}(x)$, and by Theorem 3.1

$$(12) \quad \chi_0(x, P) = \frac{-\alpha_{i,0}(x)\gamma'_i(x)}{k - \gamma_i(x)} + d_{i,0}(x) + O(k - \gamma_i(x)),$$

$$(13) \quad \chi_1(x, P) = \frac{-\gamma'_i(x)}{k - \gamma_i(x)} + d_{i,1}(x) + O(k - \gamma_i(x)),$$

$$(14) \quad d_{i,0}(x) = \alpha_{i,0}^2(x) + \alpha_{i,0}(x)d_{i,1}(x) - \alpha'_{i,0}(x).$$

Let Γ be the hyperelliptic spectral curve

$$(15) \quad w^2 = F_g(z) = z^{2g+1} + c_{2g}z^{2g} + \dots + c_0,$$

and $q = \infty \in \Gamma$, $k = \frac{1}{\sqrt{z}}$. The curve Γ has the involution

$$\sigma : \Gamma \rightarrow \Gamma, \quad \sigma(z, w) = (z, -w).$$

Let us find coefficients of the operator of order 4 corresponding to z

$$L_4 \psi = (\partial_x^4 + f_2(x)\partial_x^2 + f_1(x)\partial_x + f_0(x))\psi = z\psi.$$

From (10) it follows that the fourth derivative of ψ is

$$\psi^{(4)} = (\chi_0^2 + \chi_1\chi_0' + \chi_0(\chi_1^2 + 2\chi_1') + \chi_0'')\psi + (\chi_1^3 + 2\chi_1\chi_0' + \chi_1(2\chi_0 + 3\chi_1') + \chi_1'')\psi'.$$

With the help of (10) and the last identity we rewrite $L_4 \psi = z\psi$ in the form $P_1\psi + P_2\psi' = z\psi$, where

$$\begin{aligned} P_1 &= f_0 + f_2\chi_0 + \chi_0^2 + \chi_1\chi_0' + \chi_0(\chi_1^2 + 2\chi_1') + \chi_0'', \\ P_2 &= f_1 + f_2\chi_1 + \chi_1^3 + 2\chi_1\chi_0' + \chi_1(2\chi_0 + 3\chi_1') + \chi_1''. \end{aligned}$$

It gives

$$(16) \quad P_1 = z = \frac{1}{k^2}, \quad P_2 = 0.$$

From (11) we have

$$P_1 - \frac{1}{k^2} = \frac{f_2 + 2a_0}{k} + (f_0 + a_0(f_2 + a_0) + 2(a_1 + b_1') + a_0'') + O(k) = 0,$$

$$P_2 = (f_1 + 2(b_1 + a_0')) + O(k) = 0.$$

Hence $f_0 = a_0^2 - 2a_1 - 2b_1' - a_0''$, $f_1 = -2(b_1 + a_0')$, $f_2 = -2a_0$. If $b_1 = 0$, then the operator L_4 is self-adjoint

$$L_4 = (\partial_x^2 + V(x))^2 + W(x),$$

where $V(x) = -a_0(x)$, $W = -2a_1(x)$. If $\chi_1(x, P) = \chi_1(x, \sigma(P))$, then

$$\chi_1 = \sum_{s>1} b_{2s} k^{2s},$$

hence, L_4 is self-adjoint (the inverse is also true, see Theorem 4.1). Assume that χ_1 is invariant under σ , then by (11)–(13) we have

$$\chi_0 = -\frac{H_1(x)\gamma_1'(x)}{z - \gamma_1(x)} - \dots - \frac{H_g(x)\gamma_g'(x)}{z - \gamma_g(x)} + \frac{w(z)}{(z - \gamma_1(x)) \dots (z - \gamma_g(x))} + \kappa(x),$$

$$\chi_1(x, P) = -\frac{\gamma_1'(x)}{z - \gamma_1(x)} - \dots - \frac{\gamma_g'(x)}{z - \gamma_g(x)},$$

where $H_i(x), \kappa(x)$ are some functions. In the neighbourhood of q the function χ_0 has the expansion

$$\chi_0 = \frac{1}{k} + \kappa + \left(\gamma_1 + \dots + \gamma_g + \frac{c_{2g}}{2} \right) k + O(k^2).$$

Hence,

$$(17) \quad V = -\kappa, \quad W = -2(\gamma_1 + \dots + \gamma_g) - c_{2g}, \quad 1 \leq i \leq g.$$

Functions χ_0, χ_1 have simple poles at $P_i^\pm = (\gamma_i, \pm \sqrt{F_g(\gamma_i)})$. Denote by $\alpha_{i,0}^\pm(x), d_{i,0}(x)^\pm, d_{i,1}(x)^\pm$ coefficients of expansions of χ_0, χ_1 at P_i^\pm . From (14) we have

$$l_i^\pm = d_{i,0}^\pm - ((\alpha_{i,0}^\pm)^2 + \alpha_{i,0}^\pm d_{i,1}^\pm - (\alpha_{i,0}^\pm)') = 0.$$

From $l_i^+ - l_i^- = 0$ one can express H_i through $\gamma_1, \dots, \gamma_g$ and its derivatives. From $l_i^+ + l_i^- = 0$ one can express $\kappa(x)$ through $\gamma_i, H_i, i = 1, \dots, g$ and its derivatives. Thus, we can reduce the system (14) to the system of $g - 1$ equations on γ_i .

Let us consider two examples at $g = 1, 2$.

4.1. Dixmier operators. Let Γ be the elliptic curve

$$w^2 = F_1(z) = z^3 + c_2 z^2 + c_1 z + c_0.$$

We have

$$\chi_0 = -\frac{H_1(x)\gamma_1'(x)}{z - \gamma_1(x)} + \frac{w(z)}{z - \gamma_1(x)} + \kappa(x), \quad \chi_1(x, P) = -\frac{\gamma_1'(x)}{z - \gamma_1(x)}.$$

The coefficients of expansions of χ_0, χ_1 at P_1^\pm are

$$\alpha_{1,0}^\pm = H_1(x) \mp \frac{w(\gamma_1)}{\gamma_1'}, \quad d_{1,0}^\pm = \kappa(x) \pm \frac{F_1'(\gamma_1)}{2w(\gamma_1)}, \quad d_{1,1}^\pm = 0,$$

hence,

$$l^\pm = \kappa(x) - \left(H_1(x) \mp \frac{w(\gamma_1)}{\gamma_1'} \right)^2 + H_1'(x) \pm \frac{w(\gamma_1)\gamma_1''}{(\gamma_1')^2} = 0.$$

From $l^+ - l^- = 0$ we find $H_1(x) = -\frac{\gamma_1''(x)}{2\gamma_1'(x)}$.

From $l^+ + l^- = 0$ we get $\kappa(x) = \frac{4F_1(\gamma_1) - (\gamma_1'')^2 + 2\gamma_1'\gamma_1'''}{4(\gamma_1')^2}$.

At

$$\gamma_1 = -h_3 x - h_2, \quad F_1 = z^3 + 2h_2 z^2 + z(h_2^2 + h_1 h_3) + h_3(h_1 h_2 - h_0 h_3)$$

we have

$$\chi_0 = \frac{\sqrt{F_1(z)}}{z + h_3 x + h_2} - (h_3 x^3 + h_2 x^2 + h_1 x + h_0), \quad \chi_1 = \frac{h_3}{z + h_3 x + h_2},$$

$$V = -\kappa(x) = h_3 x^3 + h_2 x^2 + h_1 x + h_0, \quad W = 2h_3 x.$$

Thus we get the operator

$$L_4^\sharp = (\partial_x^2 + h_3x^3 + h_2x^2 + h_1x + h_0)^2 + 2h_3x.$$

At $g = 1, h_0 = h_1 = h_2 = 0, h_3 = 1$. The operator L_4^\sharp coincides with the Dixmier operator.

4.2. Spectral curves of genus two. Let Γ be a spectral curve of genus two

$$w^2 = F_2(z) = z^5 + c_4z^4 + c_3z^3 + c_2z^2 + c_1z + c_0.$$

We have

$$\begin{aligned}\chi_0 &= -\frac{H_1(x)\gamma_1'(x)}{z - \gamma_1(x)} - \frac{H_2(x)\gamma_2'(x)}{z - \gamma_2(x)} + \frac{w(z)}{(z - \gamma_1(x))(z - \gamma_2(x))} + \kappa(x), \\ \chi_1(x, P) &= -\frac{\gamma_1'(x)}{z - \gamma_1(x)} - \frac{\gamma_2'(x)}{z - \gamma_2(x)}.\end{aligned}$$

The coefficients of expansions of χ_0, χ_1 at $P_1^\pm(x), P_2^\pm(x)$ are

$$\begin{aligned}\alpha_{1,0}^\pm &= H_1(x) \mp \frac{w(\gamma_1)}{(\gamma_1 - \gamma_2)\gamma_1'}, \quad \alpha_{2,0}^\pm = H_2(x) \pm \frac{w(\gamma_2)}{(\gamma_1 - \gamma_2)\gamma_2'}, \\ d_{1,0}^\pm &= \mp \frac{w(\gamma_1)}{(\gamma_1 - \gamma_2)^2} + \kappa(x) \pm \frac{F_2'(\gamma_1)}{2w(\gamma_1)(\gamma_1 - \gamma_2)} - \frac{H_2\gamma_2'}{\gamma_1 - \gamma_2}, \\ d_{2,0}^\pm &= \mp \frac{w(\gamma_2)}{(\gamma_1 - \gamma_2)^2} + \kappa(x) \pm \frac{F_2'(\gamma_2)}{2w(\gamma_2)(\gamma_2 - \gamma_1)} + \frac{H_1\gamma_1'}{\gamma_1 - \gamma_2}, \\ d_{1,1}^\pm &= -\frac{\gamma_2'(x)}{\gamma_1(x) - \gamma_2(x)}, \quad d_{2,1}^\pm = \frac{\gamma_1'(x)}{\gamma_1(x) - \gamma_2(x)}.\end{aligned}$$

Equations $l_1^\pm = 0, l_2^\pm = 0$ have the form

$$\begin{aligned}l_1^\pm &= -\frac{1}{(\gamma_1 - \gamma_2)^2(\gamma_1')^2} (F_2(\gamma_1) + (\gamma_1 - \gamma_2)(\gamma_1')^2((H_2 - H_1)\gamma_2' \\ &\quad + (H_1^2 - \kappa - H_1')(\gamma_1 - \gamma_2)) + w(\gamma_1)(\pm 2\gamma_1'\gamma_2' \pm (2H_1\gamma_1' + \gamma_1'')(\gamma_2 - \gamma_1))) = 0, \\ l_2^\pm &= -\frac{1}{(\gamma_1 - \gamma_2)^2(\gamma_2')^2} (F_2(\gamma_2) + (\gamma_1 - \gamma_2)(\gamma_2')^2((H_2 - H_1)\gamma_1' + \\ &\quad (H_2^2 - \kappa - H_2')(\gamma_1 - \gamma_2)) + w(\gamma_2)(\pm 2\gamma_1'\gamma_2' \pm (2H_2\gamma_2' + \gamma_2'')(\gamma_1 - \gamma_2))) = 0,\end{aligned}$$

From the equations $l_1^+ - l_1^- = 0$ and $l_2^+ - l_2^- = 0$ we find

$$H_1(x) = \frac{\gamma_2'}{\gamma_1 - \gamma_2} - \frac{\gamma_1''}{2\gamma_1'}, \quad H_2(x) = -\frac{\gamma_1'}{\gamma_1 - \gamma_2} - \frac{\gamma_2''}{2\gamma_2'}.$$

From the equations $l_1^+ + l_1^- = 0$ and $l_2^+ + l_2^- = 0$ we can find $\kappa(x)$ by two ways

$$\begin{aligned}\kappa(x) &= \frac{F_2(\gamma_1) + (\gamma_1 - \gamma_2)(\gamma_1')^2((H_2 - H_1)\gamma_2' + \gamma_1(H_1^2 - H_1') - \gamma_2(H_1^2 - H_1'))}{(\gamma_1 - \gamma_2)^2\gamma_1'^2}, \\ \kappa(x) &= \frac{F_2(\gamma_2) + (\gamma_2 - \gamma_1)(\gamma_2')^2((H_1 - H_2)\gamma_1' + \gamma_2(H_2^2 - H_2') - \gamma_1(H_2^2 - H_2'))}{(\gamma_2 - \gamma_1)^2\gamma_2'^2}.\end{aligned}$$

We get the equation on γ_1 and γ_2 (see [11])

$$\begin{aligned}(18) \quad & 4((\gamma_1')^2F_2(\gamma_2) - (\gamma_2')^2F_2(\gamma_1)) - 4(\gamma_1')^4(\gamma_2')^2 + (\gamma_1 - \gamma_2)^2(\gamma_2')^2(\gamma_1'')^2 \\ & + 2(\gamma_1 - \gamma_2)(\gamma_1')^3\gamma_2'\gamma_2'' + 2(\gamma_1 - \gamma_2)\gamma_1'(\gamma_2')^2(\gamma_2'\gamma_1'' + (\gamma_2 - \gamma_1)\gamma_1''') + \\ & (\gamma_1')^2(4(\gamma_2')^4 + 6(\gamma_1 - \gamma_2)(\gamma_2')^2(\gamma_1'' + \gamma_2'') + (\gamma_1 - \gamma_2)^2(2\gamma_2'\gamma_2''' - (\gamma_2'')^2)) = 0.\end{aligned}$$

O.I. Mokhov also considered the case of self-adjoint operators of rank two, corresponding to a curve of genus two. In particular, he also reduced the equations on Tyurin parameters to one equation on two functions (see introduction in [8]).

Equation (18) has the following solution

$$\gamma_1 = \frac{1}{2}(-5h_2 - 3h_3x + p), \quad \gamma_2 = \frac{1}{2}(-5h_2 - 3h_3x - p),$$

$$p = 3\sqrt{h_2^2 - 2h_3(2h_1 - h_2x) - 3h_3^2x^2},$$

$F_2 = z^5 + 10h_2z^4 + (33h_2^2 + 21h_1h_3)z^3 + (40h_2^3 + 117h_1h_2h_3 + 27h_0h_3^3)z^2 + 4(4h_2^4 + 36h_1h_2^2h_3 + 27h_3^2(h_1^2 + h_0h_2))z + 3h_3(36h_1^2h_2h_3 + 27h_3^3 + 4h_1(4h_2^3 + 27h_0h_3^2))$. Formula (17) gives us $V = -\kappa = h_3x^3 + h_2x^2 + h_1x + h_0$, $W = -2(\gamma_1 + \gamma_2) - 10h_2 = 6h_3x$. Thus we get the operator

$$L_4^\# = (\partial_x^2 + h_3x^3 + h_2x^2 + h_1x + h_0)^2 + 6h_3x.$$

V.V. Sokolov noticed that the equation (18) has also polynomial solutions of degree two.

4.3. Hyperelliptic spectral curves. As we showed above, if χ_1 is invariant under the involution σ , then the operator L_4 is self-adjoint. S.P. Novikov has proposed the conjecture that the inverse is also true.

Theorem 4.1 (M., [12]). *The operator L_4 is self-adjoint if and only if*

$$(19) \quad \chi_1(x, P) = \chi_1(x, \sigma(P)).$$

At $g = 1$ Theorem 4.1 is equivalent to the Theorem 3.5 by Grinevich and Novikov.

Let us assume that the operator L_4 is self-adjoint $L_4 = (\partial_x^2 + V(x))^2 + W(x)$, then the functions χ_0, χ_1 have simple poles at some points

$$\left(\gamma_i(x), \pm \sqrt{F_g(\gamma_i(x))} \right), \quad 1 \leq i \leq g.$$

Theorem 4.2 (M., [12]). *If operator L_4 is self-adjoint, then*

$$\chi_0 = -\frac{1}{2} \frac{Q''}{Q} + \frac{w}{Q} - V, \quad \chi_1 = \frac{Q'}{Q},$$

where $Q = (z - \gamma_1(x)) \dots (z - \gamma_g(x))$. Functions Q, V, W satisfy the equation

$$(20) \quad 4F_g(z) = 4(z - W)Q^2 - 4V(Q')^2 + (Q'')^2 - 2Q'Q^{(3)} + 2Q(2V'Q' + 4VQ'' + Q^{(4)}),$$

where $Q', Q'', Q^{(k)}$ mean $\partial_x Q, \partial_x^2 Q, \partial_x^k Q$.

To find self-adjoint operators L_4, L_{4g+2} it is enough to solve the equation (20).

Corollary 1 *The functions Q, W, V satisfy the equation*

$$Q^{(5)} + 4VQ^3 + 2Q'(2z - 2W - V'') + 6V'Q'' - 2QW' = 0.$$

Let us substitute $z = \gamma_j$ in (20). This gives

$$V(x) = \left(\frac{(Q'')^2 - 2Q'Q^{(3)} - 4F_g(z)}{4(Q')^2} \right) \Big|_{z=\gamma_j}.$$

We get $g - 1$ equations on $\gamma_1(x), \dots, \gamma_g(x)$.

Corollary 2 *The functions $\gamma_1(x), \dots, \gamma_g(x)$ satisfy the equations*

$$(21) \quad \left(\frac{(Q'')^2 - 2Q'Q^{(3)} - 4F_g(z)}{4(Q')^2} \right) \Big|_{z=\gamma_j} = \left(\frac{(Q'')^2 - 2Q'Q^{(3)} - 4F_g(z)}{4(Q')^2} \right) \Big|_{z=\gamma_k}.$$

At $g = 2$ the equation (21) coincides with the equation (18). In [12] partial solutions of the equation (20) are found for arbitrary g . These solutions give the first examples of commuting differential operators of rank greater than one corresponding to a spectral curve of arbitrary genus.

Theorem 4.3 (M., [12]). *The operator*

$$L_4^\# = (\partial_x^2 + h_3x^3 + h_2x^2 + h_1x + h_0)^2 + g(g+1)h_3x, \quad h_3 \neq 0$$

commutes with a differential operator $L_{4g+2}^\#$ of order $4g+2$. The operators $L_4^\#, L_{4g+2}^\#$ are operators of rank two. For generic values of parameters (h_0, h_1, h_2, h_3) the spectral curve is a nonsingular hyperelliptic curve of genus g .

Operators $L_4^\#, L_{4g+2}^\#$ define commutative subalgebras in the first Weyl algebra A_1 .

5. CONCLUDING REMARKS AND QUESTIONS

1. G. Latham and E. Previato [19] proved the following statement. Let L_4 and L_6 be operators of rank two corresponding to an elliptic spectral curve. Then there are $z_0, w_0 \in \mathbb{C}$ such that

$$L_4 - z_0 = A_2T, \quad L_6 - w_0 = A_4T$$

for some operators A_2, A_4, T and such that

$$\tilde{L}_4 = TA_2 = T(L_4 - z_0)T^{-1}, \quad \tilde{L}_6 = TA_4 = T(L_6 - w_0)T^{-1}$$

are commuting self-adjoint operators of rank two. In other words, in the case of elliptic spectral curves all operators of rank two up to the conjugation by operators of the second order are self-adjoint operators. In the beginning of the 1980's O.I. Mokhov proved a similar result where T is an operator of the first order (this result is not published). It would be very interesting to check this property for operators L_4, L_{4g+2} corresponding to an hyperelliptic spectral curve of genus g .

2. The group of automorphisms of the first Weyl algebra $Aut(A_1)$ acts on the moduli spaces of operators with polynomial coefficients. For example, with the help of the automorphism

$$\varphi_1(x) = \alpha x + \beta \partial_x, \quad \varphi_1(\partial_x) = \gamma x + \delta \partial_x, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2$$

one can get from $L_4^\#, L_{4g+2}^\#$ the operators of rank 3. Another example of automorphisms are

$$\begin{aligned} \varphi_2(x) &= x + P_1(\partial_x), & \varphi_2(\partial_x) &= \partial_x, \\ \varphi_3(x) &= x, & \varphi_3(\partial_x) &= \partial_x + P_2(x), \end{aligned}$$

where P_1, P_2 are polynomials. Dixmier [24] proved that $Aut(A_1)$ is generated by φ_i . It would be very interesting to understand how $Aut(A_1)$ acts on the spectral data.

3. Theorem 4.3 means that the equation $Y^2 = X^{2g+1} + c_{2g}X^{2g} + \dots + c_0$ has nonconstant solutions $X, Y \in A_1$ for some c_i .

It is easy to see that the group $Aut(A_1)$ preserves the space of all such solutions, i.e. if (X, Y) is a solution to the polynomial equation above, with $X, Y \in A_1$, then $(\varphi(X), \varphi(Y))$ is also a solution for any $\varphi \in Aut(A_1)$. Then, a natural question is to describe the orbits of $Aut(A_1)$ in the space of solutions under the action of $Aut(A_1)$.

Yu. Berest has proposed the following conjecture: If $g > 1$, then there are only finitely many such orbits, i.e. the equation $f(X, Y) = \sum_{i,j=0}^k \alpha_{ij} X^i Y^j = 0$ with generic $\alpha_{ij} \in \mathbb{C}$ has at most finitely many solutions in A_1 up to the action of $Aut(A_1)$.

4. Let me recall the Dixmier conjecture: $End(A_1) = Aut(A_1)$. If one describe all orbits of $Aut(A_1)$ in the space of solutions for the equation $f(X, Y) = 0$, then this gives a chance to compare $End(A_1)$ and $Aut(A_1)$. For example, if there is only one orbit, then $End(A_1) = Aut(A_1)$. For this reason it is important to find all solutions $X, Y \in A_1$ for one concrete equation and to study the action of $Aut(A_1)$. For example, one can take the simplest equation $Y^2 = X^3 + 1$.

REFERENCES

- [1] G. Wallenberg, Über die Vertauschbarkeit homogener linearer Differentialausdrücke. *Arch. Math. Phys.* **4** (1903), 252–268.
- [2] J. Schur, Über vertauschbare lineare Differentialausdrücke. *Sitzungsber. der Berliner Math. Gesell.* **4** (1905), 2–8.
- [3] J.L. Burchall and T.W. Chaundy, Commutative ordinary differential operators. *Proc. London Math. Soc. Ser. 2.* **21** (1923), 420–440.
- [4] I.M. Krichever, Integration of nonlinear equations by the methods of algebraic geometry. *Functional Anal. Appl.* **11** (1977), 12–26.
- [5] I.M. Krichever, Commutative rings of ordinary linear differential operators, *Functional Anal. Appl.* **12** (1978), 175–185.
- [6] I.M. Krichever, S.P. Novikov, Holomorphic bundles over Riemann surfaces and the Kadomtsev-Petviashvili equation. I, *Functional Anal. Appl.* **12**:4 (1978), 276–286.
- [7] I.M. Krichever, S.P. Novikov, Holomorphic bundles over algebraic curves and nonlinear equations, *Russian Math. Surveys.* **35**:6 (1980), 47–68.
- [8] O.I. Mokhov, Commuting differential operators of rank 3 and nonlinear differential equations, *Mathematics of the USSR-Izvestiya.* **35**:3 (1990), 629–655.
- [9] A.E. Mironov, A ring of commuting differential operators of rank 2 corresponding to a curve of genus 2, *Sbornik: Math.* **195**:5 (2004), 711–722.
- [10] A.E. Mironov, On commuting differential operators of rank 2, *Siberian Electronic Math. Reports.* **6** (2009), 533–536.
- [11] A.E. Mironov, Commuting rank 2 differential operators corresponding to a curve of genus 2, *Functional Anal. Appl.* **39**:3 (2005), 240–243.
- [12] A.E. Mironov, Self-adjoint commuting differential operators and commutative subalgebras of the Weyl algebra, arxiv:0451666.
- [13] D. Zuo, *Commuting differential operators of rank 3 associated to a curve of genus 2*, arxiv: 1105.5774.
- [14] D. Mumford, An algebro-geometric constructions of commuting operators and of solutions to the Toda lattice equations, Korteweg-de Vries equations and related non-linear equations. In *Proc. Internat. Symp. on Alg. Geom.*, Kyoto 1977, Kinokuniya Publ. (1978) 115–153.
- [15] P.G. Grinevich, S.P. Novikov, Spectral theory of commuting operators of rank two with periodic coefficients, *Functional Anal. Appl.* **16**:1 (1982), 19–20.
- [16] P.G. Grinevich, Rational solutions for the equation of commutation of differential operators, *Functional Anal. Appl.* **16**:1 (1982), 15–19.
- [17] F. Grunbaum, Commuting pairs of linear ordinary differential operators of orders four and six, *Phys. D.* **31**:3 (1988), 424–433.
- [18] G. Latham, Rank 2 commuting ordinary differential operators and Darboux conjugates of KdV, *Appl. Math. Lett.* **8**:6 (1995), 73–78.

- [19] G. Latham, E. Previato, Darboux transformations for higher-rank Kadomtsev-Petviashvili and Krichever-Novikov equations, *Acta Appl. Math.* **39** (1995), 405–433.
- [20] O.I. Mokhov, On commutative subalgebras of Weyl algebra, which are associated with an elliptic curve. International Conference on Algebra in Memory of A.I. Shirshov (1921-1981). Barnaul, USSR, 20-25 August 1991. Reports on theory of rings, algebras and modules. 1991. P. 85.
- [21] O.I. Mokhov, On the commutative subalgebras of Weyl algebra, which are generated by the Chebyshev polynomials. Third International Conference on Algebra in Memory of M.I. Kargapolov (1928-1976). Krasnoyarsk, Russia, 23-28 August 1993. Krasnoyarsk: Inoprof, 1993. P. 421.
- [22] E. Previato, G. Wilson, Differential operators and rank 2 bundles over elliptic curves, *Compositio Math.* **81**:1 (1992), 107-119.
- [23] P. Dehornoy. Operateurs différentiels et courbes elliptiques, *Compositio Math.* **43**:1 (1981), 71-99
- [24] J. Dixmier, Sur les algèbres de Weyl, *Bull. Soc. Math. France.* **96** (1968), 209-242.
- [25] S.P. Novikov, Commuting operators of rank $l > 1$ with periodic coefficients, *Dokl. Akad. Nauk SSSR.* **263**:6 (1982), 1311–1314.

SOBOLEV INSTITUTE OF MATHEMATICS, NOVOSIBIRSK, RUSSIA,
 NOVOSIBIRSK STATE UNIVERSITY, *and*
 LABORATORY OF GEOMETRIC METHODS IN MATHEMATICAL PHYSICS, MOSCOW STATE UNIVERSITY
E-mail address: mironov@math.nsc.ru